Measure of Controllability for Actuator Placement

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A new measure of controllability for linear time invariant dynamical systems is introduced. The controllability measure is designed especially to guide the placement of actuators to control vibrating structures. An example is presented, and the design of optimal feedback control laws for each of several actuator configurations supports the practical value of the new controllability measure.

Introduction

THE problem of choosing actuator locations for the control of large flexible space structures is an important area of current research. When the fundamental role played by the actuator location is recognized, it is natural that the problem of placing actuators be considered simultaneously with the quest to define meaningful measures of controllability (MOC).

Viswanathan et al.[1] and Lindberg and Longman[2] present a definition of a degree of controllability and apply it to optimize actuator placement. For the purpose of model reduction, Moore[3] introduces an internally balanced system by using the singular values to define measures of nearness to rank deficiency of the controllabilty and observability grammians. Based upon this approach, the smallest singular value of the controllability grammian (of Moore's balanced system) can be taken as the MOC. Skelton and DeLorenzo[4] and Skelton and Chiu[5] apply input (output) cost analysis, which assigns each actuator's (sensor's) contribution in the system performance to select the best set of actuators (sensors). Hamdan and Nayfeh[6] propose a new measure of modal controllability by using the generalized angles between the left eigenvectors of the system matrix A and the columns of the input influence matrix B for the system described by the triple (A, B, C). They also show that their measure has interesting connections with Longman's degree of controllability and is also related to the singular values of Moore's balancing method.[7] Hamdan and Nayfeh's easily computed measure provides us with useful information on each mode's controllability. Our analytical and numerical studies indicate that Hamdan and Nayfeh's measure is among the most attractive of the modal controllability measures available in the literatures.

In this paper, we extend Hamdan and Nayfeh's measure by introducing a new controllability index that combines their controllability ideas with modal cost analysis.[8,9] Our index addresses simultaneously the physical importance of each mode along with its degree of controllability. To evaluate the usefulness of the proposed new index, we design two sets of controllers—one set using symmetric output feedback[10] and another set using the linear quadratic regulator approach. Each set includes controllers for 10 different configurations (that is, 10 different actuator locations for the same structure), and we compare the results of the 10 pairs of controllers to judge the merits of the proposed new index.

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Measure of Modal Controllability

Consider the linear dynamical system in the state-of-space form

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in R^n$, $u \in R^m$, and A and B are real matrices with appropriate dimensions.

It is well known that the Popov, Belevitch, and Hautus (PBH) eigenvector test[11] is useful to test the modal controllability of the system, Eq. (1). The PBH eigenvector test specifies that any column vector $\boldsymbol{b}_j$ of input matrix B cannot be orthogonal to the ith left eigenvector $\boldsymbol{q}_i$ of A if the ith mode of the system is controllable. Unfortunately, the information from the PBH eigenvector test is a binary Yes/No type. By introducing a geometrical interpretation of the PBH eigenvector test, Hamdan and Nayfeh[6] proposed ith mode's gross measures of controllability ρ_i as follows:

$$f_i^T = q_i^T B / \|q_i\|$$

$$\rho_i = \|f_i\|_2 \tag{2}$$

When we compute the measure of controllability, we should be careful dealing with the coordinate transformation. Since these measures are invariant only under orthonormal coordinate transformations, the measures should be used consistently only after all transformations, including scaling, have been carried out.

The least controllable mode is often considered to be critical for the controllability of the system (by examining the definition of the degree of controllability for modal systems[1]). Analogous to using the smallest singular value as a measure of controllability in the balancing approach,[3] we could conjecture the following rule: "The smallest value of the gross measures of modal controllability ρ_j is the controllability index for the given system." This rule, while perhaps intuitively appealing, is definitely false for most practical applications. Consider a system for which the kth mode of the system is the least controllable, then the kth mode's gross measure of controllability ρ_k (i.e., the minimum ρ_k) would be taken as the system's controllability index if we use the above rule. However, if the kth mode does not participate significantly in the important physical outputs of the system, then using only the controllability of the kth mode to characterize the system controllability is clearly not suitable. Therefore we need an index that incorporates more information to measure the controllability of the system properly. Some measure of the relative importance of each mode vis-a-vis this mode's contribution to the system performance is recommended to weight the modal controllability measures. The most appealing approach is to combine the modal cost ideas of Skelton[8] and Skelton et al.[9] with the modal controllability ideas of Hamdan and Nayfeh.[6] We develop this approach in the following sections and define a new measure of controllability.

Modal Cost Analysis

The scalar function

$$\int_0^\infty y^T(t) Q y(t)\, \mathrm{d}t$$

is widely used as a measure of system performance where the vector $y(t)$ is composed of those output error variables that are of importance to the designer. The unit impulse response with zero initial condition is also commonly used to discuss the transient response of the system. By considering the above two observations, we can take the cost function that represents the system performance as follows:

$$V = \sum_{i=1}^{m} \int_0^\infty y_d^{iT}(t) Q_v y_d^i(t)\, \mathrm{d}t$$

$$y_d^i(t) = C_d x^i(t) \tag{3}$$

where $x^i(t)$ is the response due to the unit impulse input $u_i(t) = \delta(t)$ (with $u_j(t) = 0$, $i \neq j$) applied at $t = 0$ with zero initial conditions, Q_v is a weighting matrix, and C_d is defined so that $y_d(t)$ models an important output for the design objectives.

The developments leading to a modal decomposition of this cost function have been derived by Skelton.[8] We may use the contribution of each modal state variable to the cost function as a measure of that mode's relative importance in the system performance. The cost function V and the contribution V_{x_i} of the state variable x_i in the cost function V can be computed as follows[8]:

$$V = \mathrm{tr}\{ Q_v C_d X C_d^T \} \tag{4}$$

$$V_{xi} = [X C_d^T Q_v C_d]_{ii} \tag{5}$$

where tr{ } denotes the trace (diagonal sum) of the matrix { }, and X is the controllability grammian that satisfies the following Lyapunov equation

$$XA^T + AX + BB^T = 0 \tag{6}$$

What we really need is each mode's contribution to the cost function, which is called the modal cost. When the modal coordinates are used as the state vector, it is obvious that each state's contribution becomes the modal cost. For a formulation of the system dynamics that uses physical or configuration coordinates in the state vector, on the other hand, the modal cost can be obtained via a modal coordinate transformation.

For a system having a given actuator placement configuration, we can determine each mode's gross measure of controllability and modal cost by using the results presented previously and in the current section. In the following section, we address the issue of how to combine the modal cost with the gross measure of modal controllability to define a new measure of controllability for the purpose of deciding where to place the actuators.

Output Measure of Controllability for the Second-Order System

In the study of vibrating mechanical systems, we usually encounter a system of n second-order equations of the form

$$M\ddot{x} + C\dot{x} + Kx = Du \tag{7}$$

where $x \in R^n$ and $u \in R^m$ are configuration and control vectors, respectively, M is an $n \times n$ positive-definite symmetric mass matrix, C is an $n \times n$ positive-semidefinite symmetric structural damping matrix, K is an $n \times n$ positive-semidefinite stiffness matrix, D is an $n \times m$ control influence matrix, and $(\dot{\ })$ represents differential with respect to time.

In the previous section, we mentioned that the MOC is generally variant under any coordinate transformation that is not orthogonal. We need to choose the coordinate system to discuss controllability. When modal coordinates are used, two well-known benefits are realized: 1) the computational processes become simplified, and 2) each state's contribution to the cost function corresponds to the modal cost. In addition, order-reduction procedures can be used to limit the discussion to a finite number of most important modes. We will use the modal coordinates for all subsequent discussions.

To perform the modal coordinate transformation, the following open-loop eigenvalue problem should be solved first[12]:

$$K\underline{\phi}_i = \lambda_i M \underline{\phi}_i, \qquad i = 1,2,\ldots,n \tag{8}$$

with the normalization equation

$$\underline{\phi}_i^T M \underline{\phi}_i = 1, \qquad i = 1,2,\ldots,n \tag{9}$$

We introduce the modal matrix

$$\Phi = [\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n] \tag{10}$$

and the general modal coordinate transformation is

$$x(t) = \Phi \underline{\eta}(t) \tag{11}$$

where $\underline{\eta}(t)$ is the $n \times 1$ vector of modal coordinates.

The transformed equation of motion becomes

$$\tilde{M}\ddot{\underline{\eta}} + \tilde{C}\dot{\underline{\eta}} + \tilde{K}\underline{\eta} = \tilde{D}u \tag{12}$$

where

$$\tilde{M} = \Phi^T M \Phi = I$$

$$\tilde{C} = \Phi^T C \Phi = \mathrm{diag}(2\zeta_1\omega_1, 2\zeta_2\omega_2, \ldots, 2\zeta_n\omega_n)$$

$$\tilde{K} = \Phi^T K \Phi = \mathrm{diag}(\omega_1^2, \omega_2^2, \ldots, \omega_n^2)$$

$$\tilde{D} = \Phi^T D$$

The diagonal structure of $\tilde{C}$ requires that C be a linear combination of M and K (see Ref. 13 for more general condition). We make this restriction for the sake of convenience. The results developed for computing Hamdan and Nayfeh's controllability measures are directly applicable to the second-order representation, Eq. (12). These are reported in Ref. 6.

For control applications, the system dynamics are usually modeled in first-order differential equations. Let us introduce the $2n$ dimensional modal state vector

$$z = \begin{pmatrix} \underline{\eta} \\ \dot{\underline{\eta}} \end{pmatrix} \tag{13}$$

Using Eq. (13), Eq. (12) can be written as the first-order system

$$\dot{z} = Az + Bu \tag{14}$$

where

$$A = \begin{bmatrix} 0 & I \\ -\tilde{K} & -\tilde{C} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \tilde{D} \end{bmatrix} \tag{15}$$

The right and left eigenvalue problems [associated with $z = pe^{\lambda t}$ solutions of the open-loop system in Eq. (14)] are

$$\text{Right:} \qquad \lambda_i p_i = A p_i, \qquad i = 1,2,\ldots,2n \tag{16a}$$

Left: $\lambda_i q_i = A^T q_i, \qquad i = 1,2,\ldots,2n \qquad (16b)$

where the conventional normalization[14] of the biorthogonality conditions for the eigenvectors are adopted as

$$p_i^T p_i = 1, \qquad i = 1,2,\ldots,2n \tag{17a}$$

$$q_j^T p_i = \delta_{ij}, \qquad i,j = 1,2,\ldots,2n \tag{17b}$$

so that

$$q_j^T A p_i = \lambda_i \delta_{ij}, \qquad i,j = 1,2,\ldots,2n \tag{18}$$

The gross measure of modal controllability can be obtained by using Eq. (2) with the above left eigenvectors and B matrix in Eq. (14).

To evaluate the modal cost for the system, we take the following system performance [Eq. (3)] as the cost function

$$V = \sum_{i=1}^{m} \int_0^\infty y_d^{i\,T}(t) Q_v y_d^i(t)\,\mathrm{d}t \tag{19}$$

where $y_d^i(t)$ is the output response due to the unit impulse input $u_i(t) = \delta(t)$ [with $u_j(t)=0$, $i \neq j$] applied at $t=0$ with zero initial conditions, and

$$y_d(t) = \begin{bmatrix} C_{dx} & 0 \\ 0 & C_{d\dot{x}} \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \equiv C_d \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \tag{20}$$

Note that by taking a suitable Q_v weighting matrix, $y_d(t)$ becomes a vector of physically important variables.

We adopted the weighting matrix as follows:

$$Q_v = \begin{bmatrix} Q_x & 0 \\ 0 & Q_{\dot{x}} \end{bmatrix} \tag{21}$$

By introducing the modal coordinate transformation, the vector $y_d(t)$ becomes

$$y_d(t) = C_d \begin{bmatrix} \Phi\underline{\eta}(t) \\ \Phi\dot{\underline{\eta}}(t) \end{bmatrix} = \begin{bmatrix} C_{dx}\Phi & 0 \\ 0 & C_{d\dot{x}}\Phi \end{bmatrix} \begin{bmatrix} \underline{\eta}(t) \\ \dot{\underline{\eta}}(t) \end{bmatrix} \equiv C_{d_\eta} \begin{bmatrix} \underline{\eta}(t) \\ \dot{\underline{\eta}}(t) \end{bmatrix} \tag{22}$$

Therefore matrix C_d in Eqs. (4–5) must be replaced by C_{d_η} to evaluate the total cost and modal cost in the modal coordinate system. Since modal coordinates are used, we can take advantage of the analytical solution of the $2n \times 2n$ Lyapunov equation, Eq. (6), available in Ref. 9:

$$X = \begin{bmatrix} X_{\eta\eta} & X_{\eta\dot{\eta}} \\ X_{\eta\dot{\eta}}^T & X_{\dot{\eta}\dot{\eta}} \end{bmatrix} \tag{23}$$

where the elements of $n \times n$ block matrices are

$$[X_{\eta\eta}]_{ij} = \frac{2(\zeta_i\omega_i + \zeta_j\omega_j)}{\Delta} [\tilde{D}\tilde{D}^T]_{ij}, \qquad i,j = 1,2,\ldots,n \tag{24a}$$

$$[X_{\eta\dot{\eta}}]_{ij} = \frac{\omega_i^2 - \omega_j^2}{\Delta} [\tilde{D}\tilde{D}^T]_{ij}, \qquad i,j = 1,2,\ldots,n \tag{24b}$$

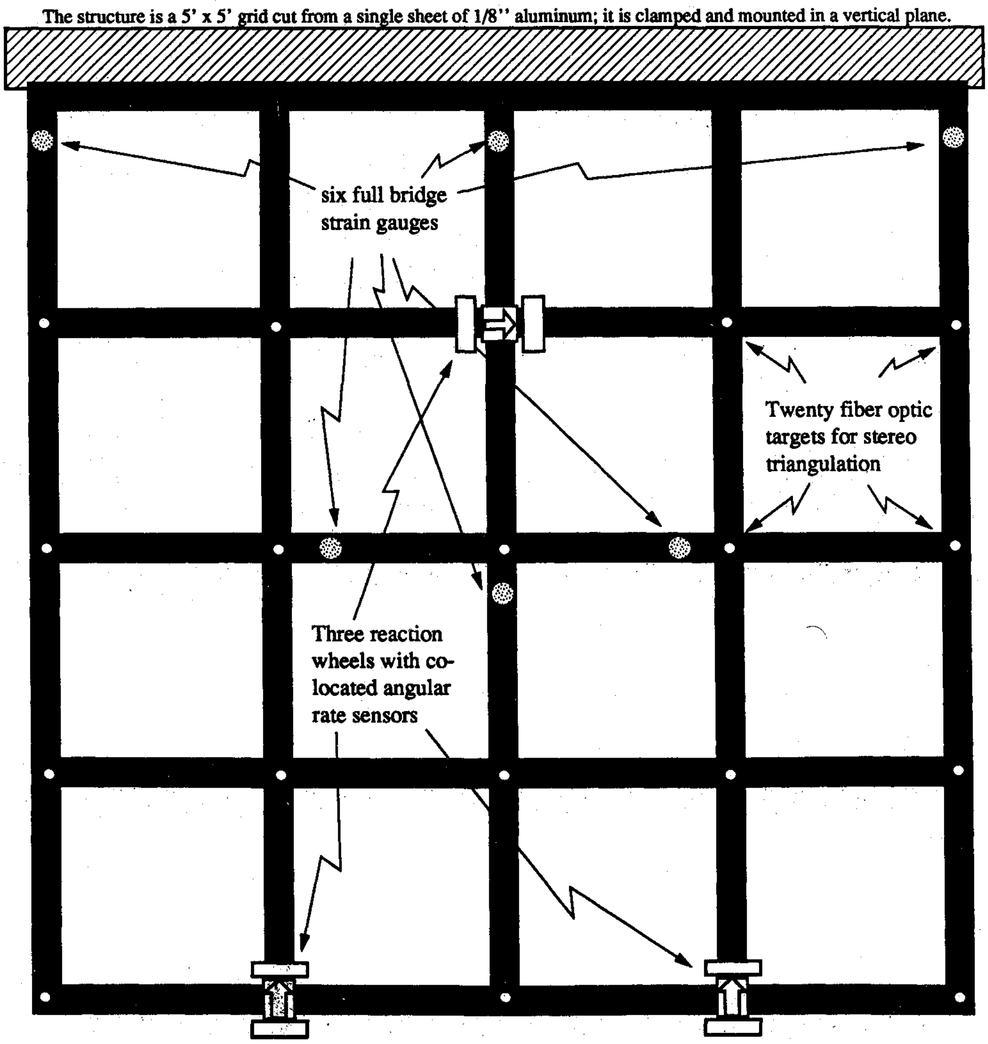

Fig. 1 Texas A&M structural control test article.

$$[X_{\dot{\eta}\dot{\eta}}]_{ij} = \frac{2\omega_i\omega_j(\zeta_i\omega_j + \zeta_j\omega_i)}{\Delta}[\tilde{D}\tilde{D}^T]_{ij}, \qquad i,j = 1,2,\ldots,n \tag{24c}$$

with

$$\Delta = (\omega_i^2 - \omega_j^2)^2 + 4\omega_i\omega_j(\zeta_i\omega_i + \zeta_j\omega_j)(\zeta_i\omega_j + \zeta_j\omega_i)$$

Note that the ith mode's modal cost consists of two parts: the modal cost of the ith mode's displacement and the modal cost of the corresponding mode's velocity, that is,

$$V_i = V_{\eta_i} + V_{\dot{\eta}_i} \tag{25}$$

where V_i is the ith mode's cost and η_i is the modal coordinate.

These V_{η_i} and $V_{\dot{\eta}_i}$ can be obtained by substituting Eqs. (21-23) into Eq. (5) as follows:

$$V_{\eta_i} = [X_{\eta\eta}\Phi^T C_{dx}^T Q_x C_{dx}\Phi]_{ii} \tag{26a}$$

$$V_{\dot{\eta}_i} = [X_{\dot{\eta}\dot{\eta}}\Phi^T C_{\dot{d}x}^T Q_x C_{\dot{d}x}\Phi]_{ii} \tag{26b}$$

By judicious selection (problem dependent, obviously) of the physically important variable vector $y_d(t)$, the modal cost V_i in the cost function V represents the contribution of the ith mode. Thus the normalized modal cost (V_i/V) provides a measure of each mode's relative importance in the system performance. We conjecture that the normalized modal cost (V_i/V) is precisely the measure of importance needed to weight the modal controllability measure of Hamdan and Nayfeh.

In view of the preceding considerations, we introduce the following new index as a measure of controllability:

$$\alpha = \sum_{i=1}^{n} \frac{V_i}{V}\rho_i^2 \tag{27}$$

where α is a new controllability index, V_i the ith mode's component cost in the cost function V, and ρ_i the ith mode's gross measure of modal controllability from all inputs [Eq. (2)]. Qualitatively, this new index represents a measure of "output controllability" that measures both modal controllability and the modal participation of all modes in the physically important cost function.

In the following section, we design two controllers (with several actuator configurations) for an example flexible structure and use the results to evaluate the utility of the new index.

Numerical Examples

To study the utility of the new index proposed in the previous section, two controllers 1) a robust output feedback controller,[10,15] and 2) a linear quadratic regulator have been designed for the example flexible structure. We compare the results of these controllers (for several actuator configurations) with the new controllability index.

We adopt as an example, a 60 degree-of-freedom model of a grid structure.[15] A finite element analysis of the grid was performed. Nodes were placed at each of the 20 joints on the grid, and each substructure of the grid was modeled using beam elements. Three degrees of freedom (one normal displacement, a transverse rotation, and a vertical rotation) appropriate for motion normal to the nominal plane of the grid were considered for each node. Figure 1 shows the flexible grid experimental configuration. The material properties for this model are listed in Table 1.

Feedback control torques are provided by three reaction wheel actuators. The actuator axes (about which control torques are applied) lie in the plane of the grid as indicated by the arrows in Fig. 1. The actuators have approximately a 60 Hz bandwidth and ±20 oz-in. (±0.1412 N-m) saturation. The grid angular displacement and velocity are measured about the same axes with solid-state sensors. These sensors have a DC to 100 Hz bandwidth accurate to ±0.001 deg and $\pm 10^{-5}$ rad/s. The particular position and orientation of the sensors and actuators shown in Fig. 1 represent one (configuration 6) of the 10 configurations that will be discussed. To avoid disruption of the symmetrical property of the grid structure, only symmetrical locations and the direction of the actuators are considered in the present discussion, the admissible actuator locations considered are the 20 grid locations, and the torque axis of each actuator is permitted to be either vertical or horizontal. The locations and the torque directions of the actuators of 10 configurations are displayed in Fig. 2. In all cases, the actuator torque axis and the active sensor axis are collocated as accurately as possible. We present only numerical studies in the present paper. A future paper will address experimental issues and further analytical/numerical/experimental results.

For high-order systems such as a flexible structure, it is usually desirable and often necessary to develop a reduced-order model to save on computational time when designing control laws. A high-order model is generally retained to verify the resulting design. Modal truncation method is used to obtain a reduced-order model for the current design process. In

Table 1 Material properties of grid model

EI	3255.2 lb × in.^{2}	9.34982 N × m^2
GJ	4943.0 lb × in.^{2}	14.18550 N × m^2
ρA	0.0000648 lb × s^2/in.^{2}	0.44678 kg/m
ρJ	0.0000217 lb × s^2	0.0000965 kg × m

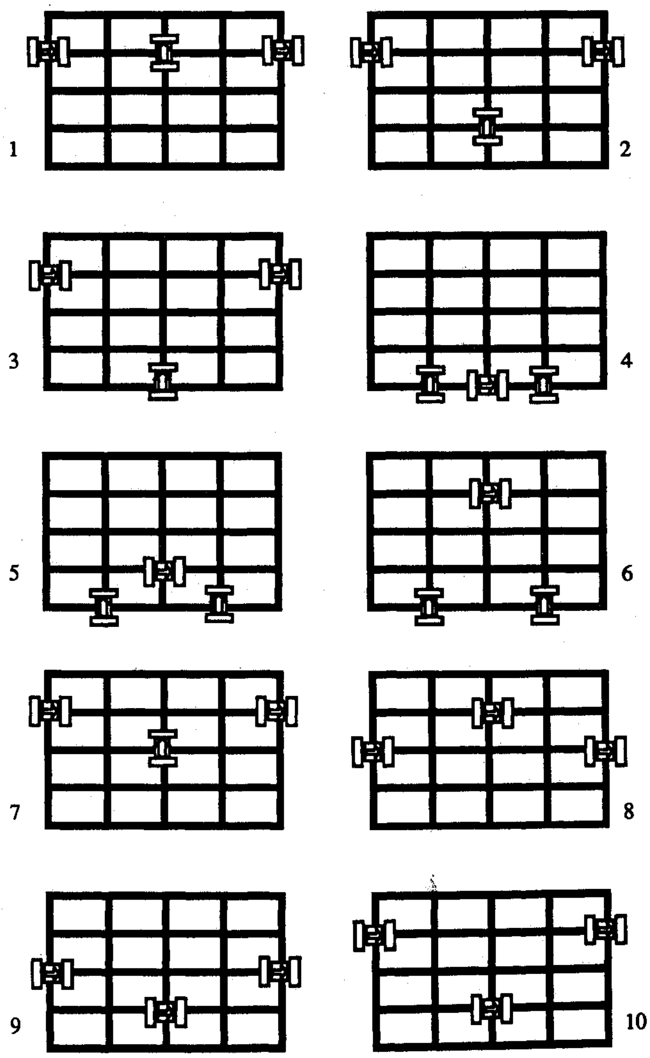

Fig. 2 Ten sets of actuator locations.

this example, we simply adopted the first 10 lowest frequency modes. The order reduction process is not central to this discussion, although it is important that a good reduced-order model be used to obtain practical results efficiently. Since the controllers are designed by using the reduced-order model, there are no additional difficulties introduced by using a reduced-order model to evaluate the new index. Some physically important variable vector is selected to construct the cost function. For this simple illustration, the normal displacement (perpendicular to the grid plane) of the lower left corner is taken to be y_d.

Control Design 1: Symmetric Output Feedback Design

For the preceding model, we apply a symmetric output feedback controller[10,15] design method to move the system's first three modes to a desired region of the left half plane and, subject to this condition, minimize a robustness measure (e.g., the sensitivity of the eigenvalues with respect to variation of uncertain parameters). For the measure of sensitivity, we use the condition number of the closed-loop eigenvector matrix.

The symmetric or structural output feedback form of the control law (collocated sensors and actuators) is given by

$$u = -(G_1\tilde{D}^T\eta_1 + G_2\tilde{D}^T\dot{\eta}_1) \tag{28}$$

where G_1 and G_2 are $m \times m$ positive definite symmetric gain matrices, and η_1 is the retained subset of modal coordinates that we are interested in the reduced-order model.

For any/all choices for the gain matrices from this stable family (i.e., the set of all positive definite G_i matrices), asymptotic stability of the closed-loop system is guaranteed in the Lyapunov sense. Of course, we do not choose the G_i at random. Positive-definite gain matrices are parameterized by introducing the Cholesky decomposition, and the gain parameter vector is defined as p_{gain}, the distinct elements of the Cholesky factors of G_1 and G_2.[10,16] With the above formulation, our gain design problem becomes a nonlinear optimization problem as follows:

$$\text{Minimize} \qquad \mathcal{K}\left[\Phi^c(p_{\text{gain}})\right] \tag{29}$$

subject to closed-loop eigenvalue placement constraints; for example, we adopt the frequency constraints

$$\omega_i^o - \Delta\omega_i \le \omega_i^c \le \omega_i^o + \Delta\omega_i, \qquad i = 1,\ldots,3$$

$$\Delta\omega_i = 1 \text{ percent of } \omega_i^o$$

and the time constant constraints

$$\omega_i^c \zeta_i^c \ge T_i, \qquad i = 1,\ldots,3$$

where

$$\omega_i^c(p_{\text{gain}}) = Im(\lambda_i)$$

$$\zeta_i^c(p_{\text{gain}}) = -\frac{Re(\lambda_i)}{|\lambda_i|}$$

and $\mathcal{K}[\Phi^c(p_{\text{gain}})]$ is the condition number of the closed-loop eigenvector matrix, the superscripts c and o denote the closed loop and the open loop, and Im and Re denote the imaginary value and real value, respectively.

For this example we take the time constants as $T_1 = 0.2$, $T_2 = 0.2$, and $T_3 = 0.25$. The above nonlinear constrained optimization problem can be solved by homotopic nonlinear programming[16,17] in conjunction with a minimum norm gain correction strategy. The computed output measures [α of Eq. (27)] of the system controllability for the 10 different configurations and the results of the controllers designed for the corresponding configurations are summarized in Table 2. The configuration number is an arbitrary reference number we assigned a priori for each configuration.

Table 2 Measure of controllability (output feedback controller)

Configuration	New index (α)	$\|\tilde{G}_1\|_f$	$\|\tilde{G}_2\|_f$	Cond. no.
9	0.00333	0.306	6.514	129.864
8	0.00234	0.142	7.434	131.611
10	0.00208	0.444	7.691	161.167
4	0.00158	1.600	7.732	128.707
5	0.00153	6.804	9.827	130.363
6	0.00130	1.595	31.143	147.774
3	0.00123	6.745	19.920	159.009
2	0.00102	13.379	16.483	140.672
7	0.00084	21.576	16.411	140.715
1	0.00078	22.160	25.379	141.402

Table 3 Measure of controllability (LQR: full-state feedback)

Configuration	New index (α)	$\|G\|_f$	Cond. no.
9	0.00333	11.2704	128.3908
8	0.00234	10.5103	127.8386
10	0.00208	22.1138	131.9951
4	0.00158	15.1570	129.2909
5	0.00153	48.8070	140.6412
6	0.00130	96.3776	144.0756
3	0.00123	48.9767	135.9563
2	0.00102	48.8298	135.6727
7	0.00084	47.4010	135.7873
1	0.00078	47.0973	135.7440

In the previous section, the output measure of controllability [Eq. (27)] is conjectured to properly weigh both modal controllability and participation in the output cost function. That means we expect the configuration possessing the largest index to be more output controllable than one having the smallest index. We further expect that more controllable configurations should require less energy to control the system, and therefore we anticipate the associated optimal controller designs to have smaller gain matrices. Based upon these heuristic observations, for example, if Eq. (27) is an appropriate measure, we should anticipate the large α designs will correspond to the smaller control gains. As we can see in Table 2, the magnitudes of the total gain norms of the first four configurations (configurations 9, 8, 10, and 4) are indeed significantly smaller than those of the remaining six configurations. In fact, designs 9, 8, 10, and 4 appear to be correctly ranked based on the fact that decreasing α corresponds to an increasing control gain norm. All of the above designs have closed-loop modal matrices with small condition number, and are therefore robust controllers. However, it is also significant that the small condition numbers are negatively well correlated with large α values. These results are consistent with our hypothesis that the new index of Eq. (27) is well correlated to the actual controllability and robustness of the system and can apparently serve as a good indication of the desirability of a given actuator configuration.

Control Design 2: Linear Quadratic Regulator Design

The symmetric output feedback controller used in controller design 1 (and implicit in Table 2) is perhaps unfamiliar to many readers. Since this is true and to confirm the usefulness of the new index, we also designed control laws for each of the 10 actuator configurations using the well-known linear quadratic regulator (LQR), and we compare the results with the new indices as well as the symmetric output feedback designs of Table 2. We adopted the following performance index for LQR:

$$J = \int_0^\infty (z^T Q z + u^T R u)\,\mathrm{d}t \tag{30}$$

where we adopted the weight matrices

$$Q = \begin{bmatrix} \Omega & 0 \\ 0 & I \end{bmatrix}, \qquad R = rI$$

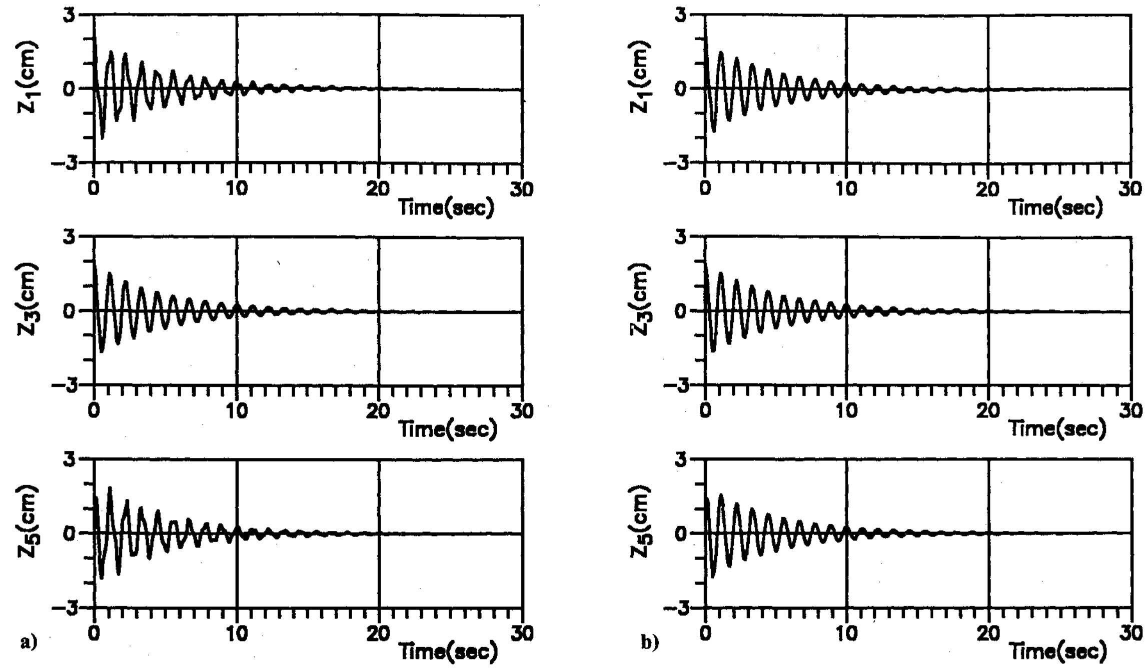

Fig. 3 Closed-loop response of the flexible grid (control design 1).

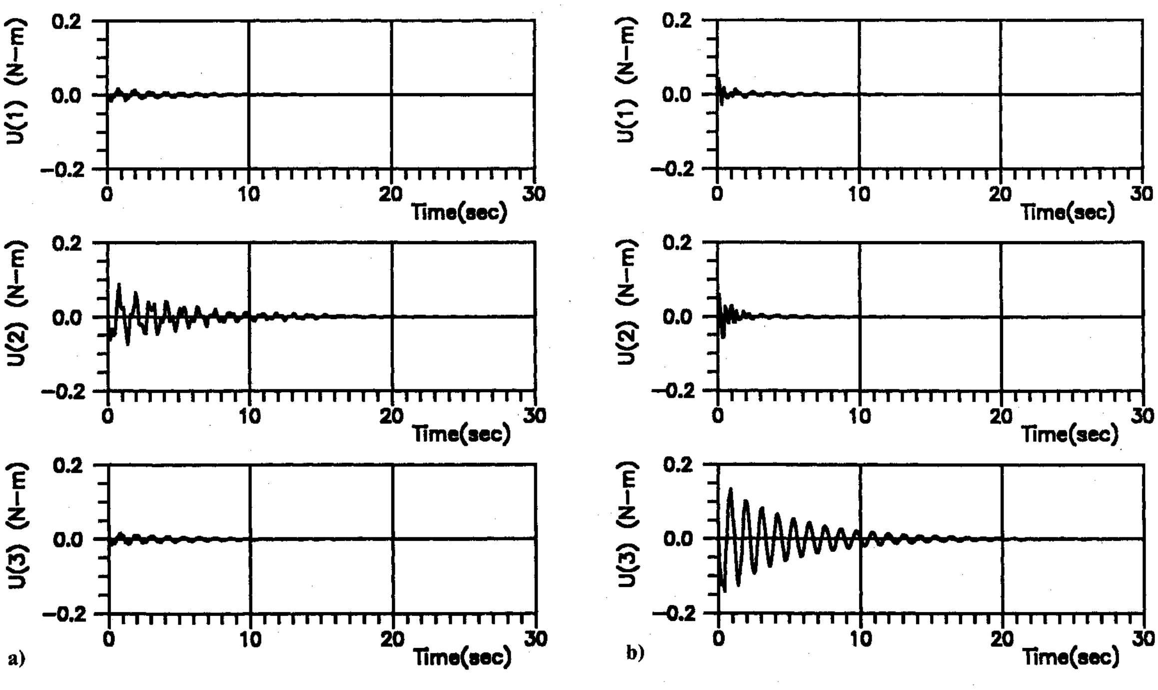

Fig. 4 Control torque input history (control design 1).

and

$$z = \begin{pmatrix} \eta_1 \\ \dot{\eta}_1 \end{pmatrix}$$

$$\Omega = \text{diag}\left[\omega_1^2, \omega_2^2, \ldots, \omega_{n_r}^2\right]$$

and the scalar r is chosen to satisfy the conditions $\omega_i^c \zeta_i^c \geq T_i$, $i = 1, \ldots, 3$ with $T_1 = 0.2$, $T_2 = 0.2$, and $T_3 = 0.25$.

The preceding performance index is an energy type, since the first term and the second term in the performance index correspond to the state energy and the control energy, respectively. Note r is a tuning factor used to place the first three closed-loop eigenvalues to satisfy the inequalities on the system time constant. Without judiciously assigning the tuning factor r, we cannot control the position of the closed-loop eigenvalues, and the closed-loop performances for the different configurations become dispersed. The results of the LQR controller designs for the 10 configurations and the corresponding configuration's new index are summarized in Table 3. Note in Table 3 that the magnitude of the gain norm of the first four configurations is again smaller than that of the remaining six configurations. Also note the norm of the 3×20 full-state feedback gain G (Table 3) cannot be directly compared to the norms of the two 3×3 output feedback gain matrices G_1 and G_2 (Table 2).

Note in Table 3 that after the first four designs (9, 8, 10, 4) the descending order of the new index is different from that of the magnitude of the output feedback controller, and is also different from that of LQR. It is obvious that the two gain optimization problems should not be expected to produce the identical results, especially since the design of Table 2 are output feedback whereas those of Table 3 are full-state feedback. Therefore we cannot expect that the new index is one-to-one mapped with the results of the two design methods studied (or any other!). In view of this, the proposed new index appears to be a remarkably good measure of controllability for the given configuration, since the results in both Tables 2 and 3 are well correlated. Notice that the first four (large α) configurations (9, 8, 10, 4) have the smallest gains in both Table 2 and 3 as compared to the other six designs.

Two of the 10 configurations—to illustrate the consequences of using relatively good and bad configurations—were adopted for further study. Especially configurations 9 and 6 were taken to study the closed-loop performance of both the symmetric output feedback control (design 1) and the LQR full-state feedback control (design 2). The closed-loop response histories of the normal displacement to the grid plane (at three nodes of the bottom of the grid structure) of two configurations due to a typical set of initial conditions are shown in Figs. 3 and 5, and the control input histories are displayed in Figs. 4 and 6. The typical set of initial conditions are constructed by using static loading such that the displacement of the lower left corner be 1 in. (2.54 cm). The static load is removed at the initial time. We have imposed control saturation bounds (20 oz-in.; 0.1412 N-m) to keep the reaction wheel speeds to modest levels. First, consider the symmetric output feedback design (control design 1). Note in Figs. 3a and 3b that the closed-loop performances of configurations 6 and 9 are almost identical. When you study Figs. 4a and 4b, however, it is evident by inspection that configuration 6 needs more control input energy. That is consistent with our expectation based upon the computed output measures of controllability. We reach the same conclusion when we compare the closed-loop

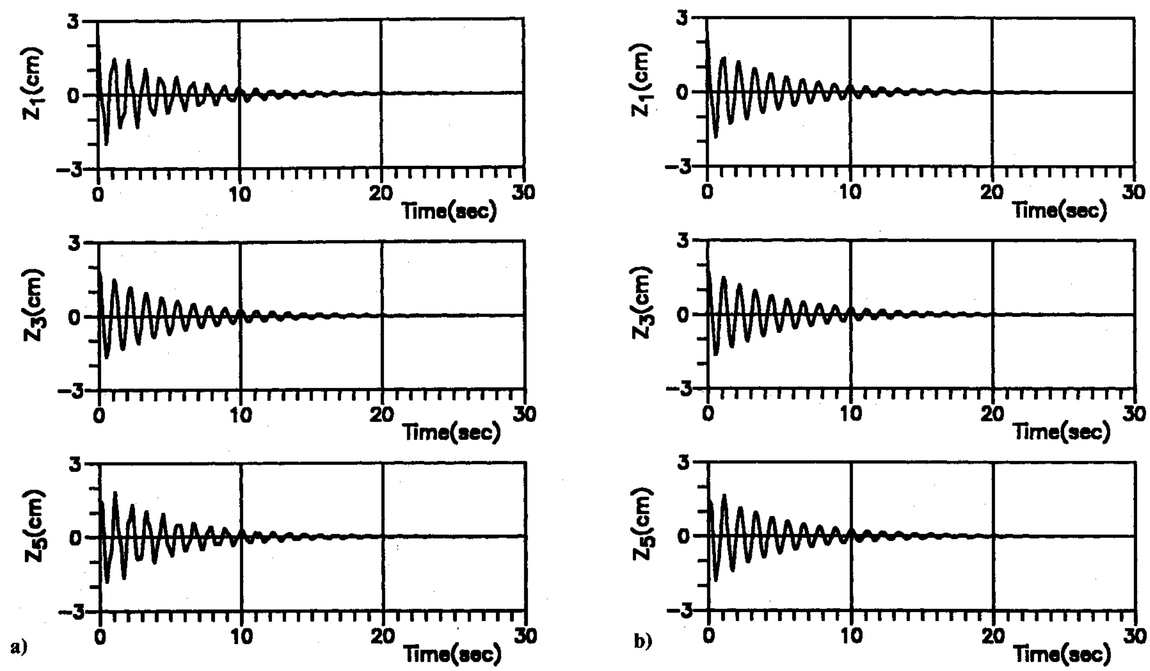

Fig. 5 Closed-loop response of the flexible grid (control design 2).

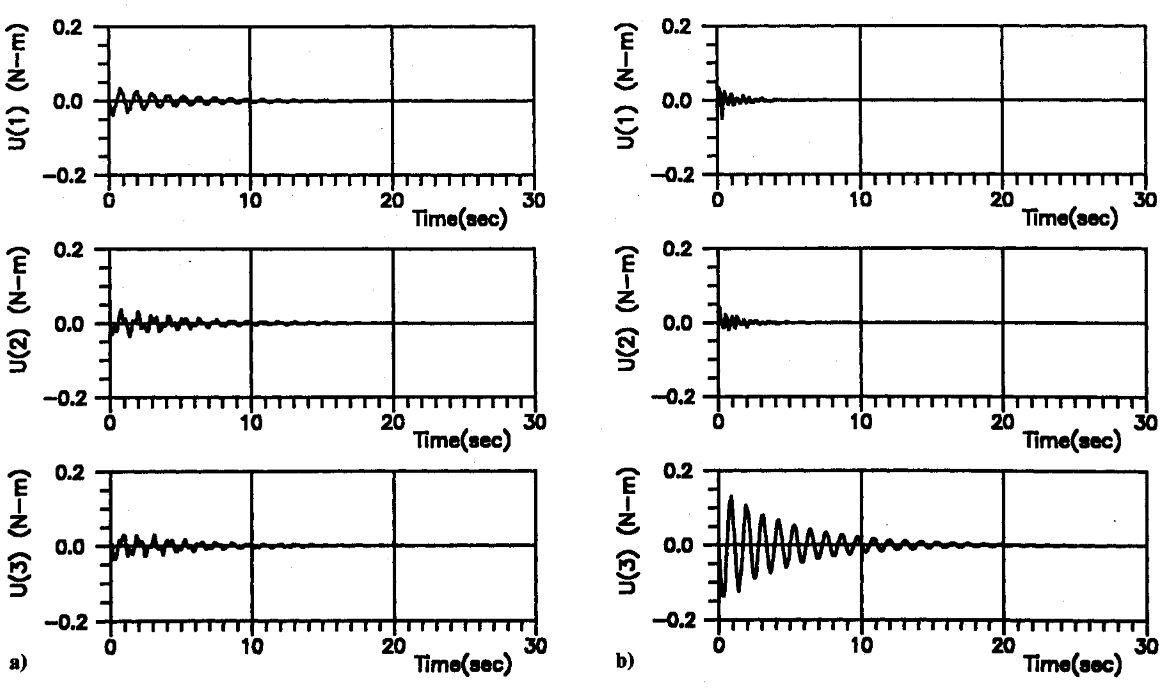

Fig. 6 Control torque input history (control design 2).

responses (Fig. 5) and the control torque histories (Fig. 6) of the LQR (control design 2). Thus we are again encouraged that the controllability measure predicts the correct trend for the controlled response with two distinct underlying controller design optimizations.

Conclusions

The present paper introduces a new measure of controllability and considers its implications for actuator placement. The proposed new index is a combination of the squares of Hamdan and Nayfeh's modal controllability measures weighted by the respective modes' contributions to a quadratic output cost function. The usefulness of this new index has been verified by comparing the results of two control design methods for a grid structure with 10 different actuator configurations.

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References

[1]Viswanathan, C. N., Longman, R. W., and Likins, P. W., "A Degree of Controllability Definition: Fundamental Concepts and Application to Modal Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 7, No. 2, 1984, pp. 222–230.

[2]Lindberg, R. E., and Longman, R. W., "Optimization of Actuator Placement via Degree of Controllability Criteria Including Spillover Considerations," AIAA/AAS Paper 82-1435, Aug. 1982.

[3]Moore, B. C., "Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction," *IEEE Transactions on Automatic Control,* Vol. AC-26, No. 1, 1981, pp. 17–32.

[4]Skelton, R. E., and DeLorenzo, M. L., "Selection of Noisy Actuators and Sensors in Linear Stochastic Systems," *Journal of Large Scale Systems, Theory and Applications,* Vol. 4, April 1983, pp. 109–136.

[5]Skelton, R. E., and Chiu, D., "Optimal Selection of Inputs and Outputs in Linear Stochastic Systems," *The Journal of the Aeronautical Sciences,* Vol. 31, No. 3, 1983, pp. 399–414.

[6]Hamdan, A. M. A., and Nayfeh, A. H., "Measure of Modal Controllability and Observability for First- and Second-Order Linear Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 3, 1989, pp. 421–428; and *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 5, 1989, p. 768.

[7]Lindner, D. L., Babendreier, J., and Hamdan, A. M. A., "Measures of Controllability and Observability and Residues," *IEEE Transactions of Automatic Control,* Vol. AC-34, No. 6, 1989, pp. 648–650.

[8]Skelton, R. E., *Dynamic System Control—Linear System Analysis and Synthesis,* Wiley, New York, 1988.

[9]Skelton, R. E., Singh, R., and Ramakrishnan, J., "Component Model Reduction by Component Cost Analysis," AIAA Guidance, Navigation, and Control Conference, AIAA Paper 88-4086, Minneapolis, MN, Aug. 1988.

[10]Junkins, J. L., and Kim, Y., "A Minimum Sensitivity Design Method for Output Feedback Controllers," AIAA Paper 90-1206, April 1990.

[11]Kailath, T., *Linear Systems,* Prentice-Hall, Englewood Cliffs, NJ, 1980.

[12]Meirovitch, L., *Computational Methods in Structural Dynamics,* Sijhoff & Noordhoff, Rockville, MD, 1980.

[13]Caughey, T. K., "Classical Normal Modes in Damped Linear Dynamic Systems," *Journal of Applied Mechanics,* Vol. 27, June 1960, pp. 269–271.

[14]Lim, K. B., Junkins, J. L., and Wang, B. P., "Re-examination of Eigenvector Derivatives," *Journal of Guidance, Control, and Dynamics,* Vol. 10, No. 6, 1987, pp. 581–587.

[15]Creamer, N. G., and Junkins, J. L., "A Pole Placement Technique for Vibration Suppression of Flexible Structures," AIAA/AAS Astrodynamic Conference, AIAA Paper 88-4254, Minneapolis, MN, Aug. 1988.

[16]Junkins, J. L., and Kim, Y., "First and Second Order Sensitivity of the Singular Value Decomposition," *Journal of the Astronautical Sciences*, Vol. 38, No. 1, 1990, pp. 69–86.

[17]Dunyak, J. P., Junkins, J. L., and Watson, L. T., "Robust Nonlinear Least Square Estimation Using the Chow-Yorke Homotopy Method," *Journal of Guidance, Control, and Dynamics,* Vol. 7, No. 6, 1984, pp. 752–755.